Exponentiated Generalized Exponential Geometric Distribution: Model, Properties and Applications

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Article History: *Received 25 Nov. 2021; Reviewed 28 Dec. 2021; Revised 5 Feb. 2022; Accepted 18 March. 2022.*

Abstract

In this article, a new distribution called Exponentiated Generalized Exponential Geometric Distribution is formulated. We have derived some important mathematical properties like hazard function, probability density function, survival function, quantiles, the measures of skewness based on quartiles and coefficient of kurtosis based on octiles. To estimate the parameters of the new distribution, we have applied the three commonly used estimation methods namely maximum likelihood estimation (MLE), least-square (LSE) method and Cramer-Von-Mises (CVM) method. We have used R programming as well as analytical methods for data analysis.For model validation, we have used different information criteria as Akaike's information criteria, and Bayesian information criteria (BIC) etc. For the assessment of potentiality of the new distribution, we have considered a real dataset and the goodness-of-fit attained by proposed distribution is compared with some competing distributions. It is found that the proposed model fits the data very well and more flexible as compared to some other models.

Keywords: *Goodness of fit, Hazard function, least square Estimation, Survival function.*

Introduction

Several new classes of models have been introduced grounded in the simple exponential distribution, having extremely wide used lifetime distribution for modeling numerous survival problems during the recent time. Although there are different ideas and purpose of deriving new models, the main idea behind generation of new model is to propose lifetime distributions which can accommodate practical as well as theoretical applications for the non-constant hazard functions, presenting monotone shapes. The exponential distribution generally does not provide a reasonable parametric fit for such practical applications. A lifetime distribution with decreasing failure rate explains a variation of the Exponential distribution and the Exponential geometric (EG) distribution, with decreasing hazard function (Adamidis and Loukas, 1998). Generalized Exponential distributions can accommodate data with increasing and decreasing hazard functions (Gupta and Kundu, 1999). A new lifetime distribution is modification of the Exponential distribution having decreasing hazard function (Kus,2007). A new lifetime distribution is

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generalized by including a power parameter deriving new distribution called "A generalization of the Exponential-Poisson distribution" (Barreto-Souza and Cribari-Neto, 2009).This distribution has increasing, decreasing and uni-modal hazard functions. Another model called the complementary exponential power lifetime model describesabout the complementary Exponential power distribution developed based on real data in actuarial science, engineering, economics and applied scienceby exponentiating the exponential power distribution(Barriga et al.,2011). In literature, some familiar distributions have been derived in different areas as: Geometric Exponential Poisson G family (Nadarajah et al., 2013), Exponentiated Exponential Poisson G family (Ristić& Nadarajah (2014)), Kumaraswamy Poisson- G family (Ramos et al., 2015), Exponentiated Generalized-G Poisson family (Aryal& Yousuf, 2017), Poisson Exponential-G family (Reyad et al., 2020).

In this paper, we have proposed a new lifetime distribution extending the exponential geometric distribution by compounding the exponentiated exponential distribution. The proposed distribution is named as Exponentiated generalized exponential geometric (EGEG). The proposed model is four parameter univariate continuous distributions. In Model formulation section, we introduce the new EGEG distribution and present some of its properties. In addition, we derive the expressions for the hazard rate function, kurtosis, probability density function, quantile function and survival function. Also, in this section we have presented the inferential procedure. We discuss some basic properties, In Methods of Parameter Estimation section; we have presented the methods of parameter estimation. In Data analysis and application section; we have taken a real data set on which proposed model is applied. Here, we have estimated the parameters of the model. In Model comparison section, we have fitted the EGEG model and made comparison with the fits of several usual lifetime distributions which can prove its relative superiority. In final section, Conclusions and comments are included.

Model Formulation

In recent years, several statisticians have been attracted by new generated families of continuous distributions to develop new models. Many of the lifetime testing problems can be solved by the exponential distribution.For analysis of life time data, fitting of data by statistical modeling is essential. Although in real life, a failure rate is not always fixed, solutions given by exponential distribution can be easily justified under assumption of constant failure rate. Since new distributions gives better fit to recent complex data, so four parameters Exponentiated Generalized Exponential Geometric Distribution (*EGEG)* has been derived. The exponential distribution has been derived in many ways to get new distribution. In literature, many families of distributions haves been derived. Alzaatreh et al. (2013), has introduced Beta exponential - X family which has following cumulative density function (CDF) and probability density function (PDF),

$$F(x) = 1 - I_{[1-F(x)]^{\lambda}}[\lambda(\beta-1)+1,\alpha] \tag{1}$$

$$f(x) = \frac{\lambda}{B(\alpha,\beta)} g(x)[1-G(x)]^{\lambda\beta-1}[1-\{1-G(x)\}]^{\lambda}]^{\alpha-1} \tag{2}$$

Where, I denote incomplete beta function. For $\beta = 1$, above CDF and PDF reduces to Exponentiated Generalized (EG) class of distribution with CDF and PDF as,

$$F(x) = [1-[1-G(x)]^{\lambda}]^{\alpha} \tag{3}$$

$$and, f(x) = \alpha\lambda g(x)[1-G(x)]^{\lambda-1}[1-\{1-G(x)\}]^{\lambda}]^{\alpha-1} \tag{4}$$

Here, we have used CDF of Exponential Geometric distribution function G(x) as the base line distribution function having CDF and PDF as

$$G(x) = \frac{1-e^{-\beta x}}{1-(1-\theta)e^{-\beta x}} \tag{5}$$

$$g(x) = \frac{\theta\beta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}^{2}} \tag{6}$$

Substituting the density function g(x) in density function of Exponentiated Exponential X family (3), we get a new density function named as Exponentiated Generalized Exponential Geometric (EGEG) distribution. The distribution function and density function of proposed model EGEG is given as

$$F(x) = \left[1-\left\{\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right\}^{\lambda}\right]^{\alpha} \tag{7}$$

$$f(x) = \frac{\alpha\lambda\beta\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}^{2}}\left[\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right]^{\lambda-1}\left[1-\left\{\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right\}^{\lambda}\right]^{\alpha-1} \tag{8}$$

Exponentiated Generalized Exponential Geometric Distribution

Let, X is non negative continuous random variable such that X~ $EGEG(\alpha,\beta,\lambda,\theta)$ with density function

$$f(x) = \frac{\alpha\lambda\beta\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}^{2}}\left[\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right]^{\lambda-1}\left[1-\left\{\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right\}^{\lambda}\right]^{\alpha-1}$$

where, $x > 0$, α, β, λ, and $\theta > 0$. α and λ shape parameters. β and θ are scale parameters.

Special Cases:

Let *X* denotes the non-negative random variable with pdf given in equation (8). We can define some other sub models from proposed model as

1. For $\alpha=1, \lambda=1$ and $\theta=1$, EGEG reduces to Exponential distribution (1) as

$$F(x)=1-e^{-\beta x}$$

2. For $\alpha=1$ and $\lambda=1$ proposed model reduces to Exponential Geometric distribution (2) as

$$F(x)=\frac{1-e^{-\beta x}}{1-(1-\theta)e^{-\beta x}}$$

3. For $\lambda=1$, proposed model reduces to Exponentiated Exponential Geometric Distribution (3) as,

$$F(x)=\left[\frac{1-e^{-\beta x}}{1-(1-\theta)e^{-\beta x}}\right]^{\alpha}$$

The density plot of the proposed model EGEG at various values of α and λ keeping β=4 and θ =0.9 is shown in figure 1. Density plot shows that the curve increases at initial value of X and then decreases gradually indicating that the proposed model is unimodal. Here it is clear that as values of α and λ varies; the shape of the curve varies indicating α and λ as the shape parameters. The probability density function of EGEG is decreasing for 0<α<1 and unimodal for α>0. From density plot it is clear that density plot of EGEG can take different shape.

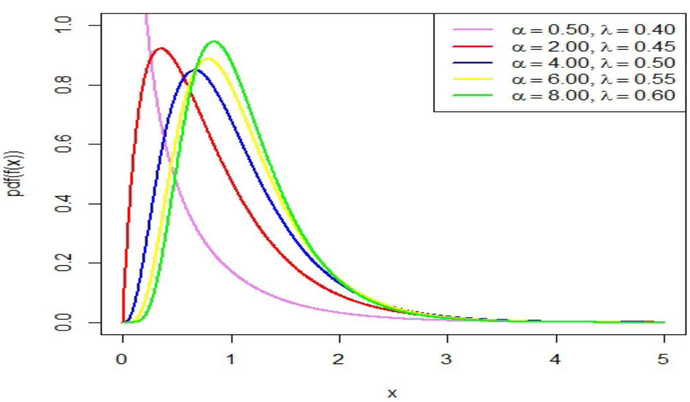

Figure1. Density plot for different values of α *and* λ *at* $\beta=4$ and θ=0.9

Statistical properties

Major characteristics of the proposed model EGEG are derived in this section.

Survival Function

The survival function is defined as the probability of an event not failing before specified time t. Survival function of EGEG is given as

$$S(x)=1-\left[1-\left\{\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right\}^{\lambda}\right]^{\alpha} \tag{9}$$

Hazard function

The hazard function is the defined as the instant failure rate at a given time t. The hazard function h(x) of the proposed model is given as

$$h(x)=\frac{\alpha\lambda\beta\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}^{2}}\left[\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right]^{\lambda-1}\left[1-\left\{\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right\}^{\lambda}\right]^{\alpha-1}$$

$$\left[1-\left[1-\left\{\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right\}^{\lambda}\right]^{\alpha-1}\right]^{-1} \tag{10}$$

The plot of the hazard rate function for different values of α and λ at $\beta=4$ and θ=0.9 is given in figure 2. The proposed distribution exhibits monotonically increasing, monotonically decreasing upside- down bathtub hazard rate. The model does not contain constant hazard rate. The hazard rate function given in equation (10) is decreasing for 0<α<1 and upside bathtub for α>0.

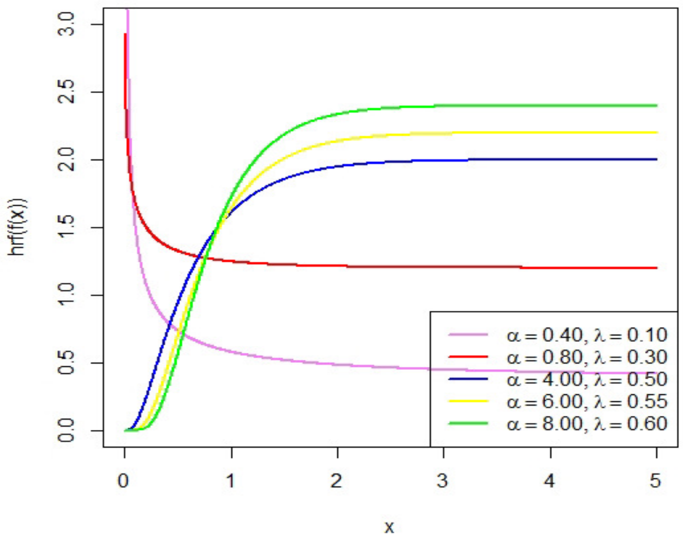

Figure2. The hazard rate function for $\beta = 4$ and θ=0.9 *at different values of* α *and* λ .

Asymptotic properties

Here are interested to observe the asymptotic properties to check whether the proposed model is uni-modal or not. For this we have found the limiting values of density function f(x) in equation (8) at x=0 and x= α That is, for $x \to 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{\alpha\lambda\beta\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}^2}\left[\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right]^{\lambda-1}\left[1-\left\{\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right\}^{\lambda}\right]^{\alpha-1}\right]$$
$$=0$$

Similarly for x $\to\propto$

$$\lim_{x\to\propto} f(x) = \lim_{x\to\propto}\left[\frac{\alpha\lambda\beta\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}^2}\left[\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right]^{\lambda-1}\left[1-\left\{\frac{\theta e^{-\beta x}}{\left\{1-(1-\theta)e^{-\beta x}\right\}}\right\}^{\lambda}\right]^{\alpha-1}\right]$$
$$=0$$

Since the limiting values of f(x) for $x \to 0$ and for x $\to \propto$ are 0 confirms that the proposed model EGEG is uni-modal.

Quantile function

The quantile function is the theoretical aspect of the probability theory.It is used for measuring statistical measures likeskewness, kurtosis and median. It is also useful in generating the random variables. Quantile function can be used as the alternative function of PDF and CDF. Quantile function can be obtained by

$$Q(p) = F^{-1}(p)$$

Quantile function for proposed model is as

$$x_p = -\frac{1}{\beta}\log\left[\frac{\left(1-p^{1/\alpha}\right)^{1/\lambda}}{\left\{\theta+(1-\theta)\left(1-p^{1/\alpha}\right)^{1/\lambda}\right\}}\right] \; ; 0<p<1 \tag{11}$$

The random Deviation Generation

Let U be the uniform (0, 1) random variable with c.d.f. $F(\bullet)$. Let, $F^{-1}(\bullet)$ exist then random number deviate can be generated from $EGEG(\alpha,\beta,\lambda,\theta)$ using expression

$$x = -\frac{1}{\beta}\log\left[\frac{\left(1-u^{1/\alpha}\right)^{1/\lambda}}{\left\{\theta+(1-\theta)\left(1-u^{1/\alpha}\right)^{1/\lambda}\right\}}\right] \; ; 0<u<1 \tag{12}$$

If we put u = 0.5 in (12) then median will be obtained.

Mode

Mode is the most repetitive value of the given PDF. To find the value of mode, the necessary and sufficient condition are; $\frac{df(x)}{dx} = 0$ and $\frac{d^2 f(x}{dx^2} < 0$. By using this necessary and sufficient condition; we get,

Solution of equation (13)cannot be found analytically as it is a nonlinear equation. Also, it cannot be found numerically by using Newton - Raphson method.

Skewness and Kurtosis

Skewness and kurtosis are the measure that describes the nature of distribution. Bowley's skewness takes the form

$$S_k = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)} \tag{14}$$

Moors' kurtosis based on Octiles is

$$K_u = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)} \tag{15}$$

where Q (.) is the quantile function. Statistical measure of the proposed model is calculated generating 100 random samples from equation (12) taking initial values $\alpha = 15.3, \beta = 1.99, \lambda = 0.58$ and θ=0.086. Using the generated valuesmedian,mean, mode, standard deviation, skewness and kurtosis are calculated to illustrate the characteristics of the proposed model.

Table 1

Mean, Median, Mode, SD, Skewness and Kurtosis of EGEG

α	β	λ	θ	Mean	Median	Mode	SD	Skewness	Kurtosis
15.3	1.99	0.58	0.086	1.7352	1.5022	1.0362	0.921134	1.248300	4.60600
15.4	1.99	0.58	0.086	1.7440	1.5113	1.0459	0.921820	1.246000	4.60040
15.5	1.99	0.58	0.086	1.7453	1.5127	1.0475	0.921967	1.245590	4.599335
15.5	2.10	0.58	0.086	1.6539	1.4335	0.9927	0.873673	1.245580	4.599335
15.5	2.20	0.58	0.086	1.5787	1.3683	0.9475	0.833961	1.245570	4.599335
15.5	2.30	0.58	0.086	1.5101	1.3088	0.9062	0.797702	1.351448	4.897776
15.5	2.30	0.59	0.086	1.4694	1.2342	0.7638	0.782006	1.351448	4.619760
15.5	2.30	0.60	0.086	1.4301	1.2342	0.8424	0.766754	1.261649	4.641028
15.5	2.30	0.61	0.086	1.3924	1.1989	0.8119	0.751926	1.269955	4.663131
15.5	2.30	0.62	0.100	1.4153	1.2264	0.8486	0.742366	1.257845	4.635275
15.5	2.30	0.62	0.120	1.4880	1.3013	0.9279	0.747586	1.235625	4.582250
15.5	2.30	0.62	0.160	1.5502	1.3652	0.9952	0.751478	1.218953	4.543713
15.5	2.30	0.62	0.160	1.6050	1.4210	1.0530	0.754495	1.205960	4.514437
15.5	2.30	0.62	0.180	1.6529	1.4702	1.1048	0.756904	1.195539	4.491443

Here, values of standard deviation are increasing as values of α, and θ are increasing and values of λ and β are decreasing. Also, different values of mean, median and mode show that the curve is not symmetrical and is not normal i.e., model is uni-modal with lack of symmetry and non-normal.

Some useful Expansion

Using the generalized binomial series, following distribution is derived. For $|z|<1, n>0,$ we have

$$(1-z)^n = \sum_{i=0}^{\infty}(-1)^i \binom{n}{i} Z^i ; n>0$$

and $$(1-z)^{-n} = \sum_{i=0}^{\infty}\binom{i+n-1`}{n-1} \quad Z^i; \quad n>0$$

For an exponential function, the power series expansion is given as;

$$e^{-az} = \sum_{j=0}^{\infty}\frac{(-1)^j (az)^j}{j!}$$

Using the above expansions, the pdf and c.d.f. of the EGEG can be expressed as

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{ij} e^{-\beta(\lambda i+\lambda+j)x} \qquad (*)$$

Where, $$\varphi_{ij} = (-1)^i \binom{\alpha-1}{i} \lambda\beta\lambda\theta^{\lambda i+\lambda}(1-\theta)^j \binom{\lambda i+\lambda+j+1}{\lambda i+\lambda}$$

and $$F(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\phi_{ij} e^{-2\beta xi}$$

Where $$\phi_{ij} = \binom{\alpha}{i}\binom{j+i-1}{i-1}(-1)^i \theta^j (1-\theta)^i$$

Moments

Moments are the quantitative measurements of the distribution in the form of function that describes the characteristics of the probability distributions. The r^{th} raw moment μ'_r of the proposed distribution $X \sim EGEG(\alpha,\beta,\lambda,\theta)$ is given as

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x)dx$$

Using equation (*), we can get r^{th} raw moment as

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{ij}\left(\frac{1}{\beta(\lambda i+\lambda+j)}\right)^r \Gamma(1+r)$$

When r =1 then mean of the EGEG will be as

$$\mu'_1 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{ij}\left(\frac{1}{\beta(\lambda i+\lambda+j)}\right)^r \Gamma(2)$$

Similarly, when r=2 then the second order raw moment of EGEG will be as

$$\mu'_2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{ij}\left(\frac{1}{\beta(\lambda i+\lambda+j)}\right)^r \Gamma(3)$$

Variance of the proposed model can be obtained by using relation $Var(X) = \mu'_2 - (\mu'_1)^2$. The lower incomplete moments $\varphi_s(t)$ is given by

$$\varphi_s(t) = \int_0^t x^s f(x)dx$$

By using density function f(x) and the lower incomplete gamma function, $\gamma(s,t) = \int_0^t x^{s-1}e^{-x}dx$, we get lower incomplete moment $\varphi_s(t)$ as;

$$\varphi_s(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{ij}\left(\frac{1}{\beta(\lambda i+\lambda+j)}\right)^{s+1} \Gamma(s+1, \quad \beta t(\lambda i+\lambda+j))$$

Methods of Parameter Estimation

Parameters of the new distribution are estimated by applying the three commonly used estimation method namely Cramer-Von-Mises (CVM), Least-square (LSE) methods and Maximum likelihood estimators (MLE).

Maximum Likelihood Estimation (MLE)

In this section, we have presented the ML estimators (MLE's) of the EGEG distribution. Let $\underline{x} = \left(x_1,\ldots,x_n\right)$ be a random sample of size 'n' from $EGEG(\alpha,\beta,\lambda,\theta)$ then the log likelihood function can be written as,

$$\ell(\alpha,\beta,\lambda,\theta \mid \underline{x}) = n\ln(\alpha\beta\lambda\theta) - \beta\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n}\ln\left\{1-(1-\theta)e^{-\beta x_i}\right\} +$$

$$(\lambda-1)\sum_{i=1}^{n}\ln\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right] + (\alpha-1)\sum_{i=1}^{n}\ln\left[1-\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{\lambda}\right] \tag{16}$$

After differentiating (16) with respect to parameters α, β, λ and θ, we get

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left[1-\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{\lambda}\right]$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i - 2(1-\theta)\sum_{i=1}^{n} x_i e^{-\beta x_i}\left\{1-(1-\theta)e^{-\beta x_i}\right\}^{-1}$$

$$-(\lambda-1)\theta\sum_{i=1}^{n} x_i e^{-\beta x_i}\left\{1-(1-\theta)e^{-\beta x_i}\right\}^{-2}\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{-1}$$

$$+(\alpha-1)\lambda\theta\sum_{i=1}^{n} x_i e^{-\beta x_i}\left[1-\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{\lambda}\right]^{-1}\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{\lambda-1}\left\{1-(1-\theta)e^{-\beta x_i}\right\}^{-2}$$

$$\frac{\partial \ell}{\partial \theta}=\frac{n}{\theta}-2\sum_{i=1}^{n} e^{-\beta x_i}\left\{1-(1-\theta)e^{-\beta x_i}\right\}^{-1}+\sum_{i=1}^{n} e^{-\beta x_i}(1-e^{-\beta x_i})\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{-1}$$

$$-\lambda(\alpha-1)\sum_{i=1}^{n} e^{-\beta x_i}\left(1-e^{-\beta x_i}\right)\left[1-\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{\lambda}\right]^{-1}\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{\lambda-1}$$

$$\frac{\partial \ell}{\partial \lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]-\lambda(\alpha-1)\sum_{i=1}^{n}\left[1-\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]^{\lambda}\right]^{-1}$$

$$\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]\ln\left[\frac{\theta e^{-\beta x_i}}{\left\{1-(1-\theta)e^{-\beta x_i}\right\}}\right]$$

By setting $\frac{\partial \ell}{\partial \alpha}=\frac{\partial \ell}{\partial \beta}=\frac{\partial \ell}{\partial \lambda}=\frac{\partial \ell}{\partial \theta}=0$ and solving them for α, β, λ and θ we get the ML estimators of the $EGEG(\alpha,\beta,\lambda,\theta)$ distribution. But normally, it is not possible to solve non-linear equations (16) so with the aid of suitable computer software one can solve them easily. Let $\underline{\Theta}=(\alpha,\beta,\lambda,\theta)$ denote the parameter vector of $EGEG(\alpha,\beta,\lambda,\theta)$ and the MLE of $\underline{\Theta}$ as $\hat{\underline{\Theta}}=(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$, then the asymptotic normality results in, $\left(\hat{\underline{\Theta}}-\underline{\Theta}\right)\rightarrow N_3\left[0,\left(I(\underline{\Theta})\right)^{-1}\right]$ where $I\left(\underline{\Theta}\right)$ is the Fisher's information matrix given by,

$$I(\underline{\Theta})=-\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\theta\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\theta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\theta^2}\right) \end{pmatrix}$$

In practice, we don't know $\underline{\Theta}$ hence it is useless that the MLE has an asymptotic variance $\left(I\left(\underline{\Theta}\right)\right)^{-1}$. Hence,by plugging in the estimated value of the parameters, we approximate the asymptotic variance. An estimate of the information matrix $I\left(\underline{\Theta}\right)$ given by the observed fisher information matrix $O(\underline{\Theta})$ as

$$O(\underline{\Theta}) = -\begin{pmatrix} \left(\frac{\partial^2 l}{\partial \alpha^2}\right) & \left(\frac{\partial^2 l}{\partial \alpha \partial \beta}\right) & \left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) & \left(\frac{\partial^2 l}{\partial \alpha \partial \theta}\right) \\ \left(\frac{\partial^2 l}{\partial \beta \partial \alpha}\right) & \left(\frac{\partial^2 l}{\partial \beta^2}\right) & \left(\frac{\partial^2 l}{\partial \beta \partial \lambda}\right) & \left(\frac{\partial^2 l}{\partial \beta \partial \theta}\right) \\ \left(\frac{\partial^2 l}{\partial \lambda \partial \alpha}\right) & \left(\frac{\partial^2 l}{\partial \lambda \partial \beta}\right) & \left(\frac{\partial^2 l}{\partial \lambda^2}\right) & \left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) \\ \left(\frac{\partial^2 l}{\partial \theta \partial \alpha}\right) & \left(\frac{\partial^2 l}{\partial \theta \partial \beta}\right) & \left(\frac{\partial^2 l}{\partial \theta \partial \lambda}\right) & \left(\frac{\partial^2 l}{\partial \theta^2}\right) \end{pmatrix}_{\big|(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})} = -H\left(\underline{\Theta}\right)_{\big|(\underline{\Theta}=\hat{\underline{\Theta}})}$$

where H is the Hessian matrix.

We use the Newton-Raphson algorithm to maximize the likelihood estimator which produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H\left(\underline{\Theta}\right)_{\big|(\underline{\Theta}=\hat{\underline{\Theta}})}\right]^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{cov}(\hat{\alpha},\hat{\theta}) \\ \mathrm{cov}(\hat{\beta},\hat{\alpha}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta},\hat{\lambda}) & \mathrm{cov}(\hat{\beta},\hat{\theta}) \\ \mathrm{cov}(\hat{\lambda},\hat{\alpha}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) & \mathrm{var}(\hat{\lambda}) & \mathrm{cov}(\hat{\lambda},\hat{\theta}) \\ \mathrm{cov}(\hat{\theta},\hat{\alpha}) & \mathrm{cov}(\hat{\theta},\hat{\beta}) & \mathrm{cov}(\hat{\theta},\hat{\lambda}) & \mathrm{var}(\hat{\theta}) \end{pmatrix} \quad (17)$$

Hence from the asymptotic normality of MLEs, approximate *100(1-a) %* confidence intervals for α, λ and θ can be constructed as,

$$\hat{\alpha} \pm Z_{a/2}\sqrt{\mathrm{var}(\hat{\alpha})}\,,\ \hat{\beta} \pm Z_{a/2}\sqrt{\mathrm{var}(\hat{\beta})},\ \hat{\lambda} \pm Z_{a/2}\sqrt{\mathrm{var}(\hat{\lambda})} \text{ and } \hat{\theta} \pm Z_{a/2}\sqrt{\mathrm{var}(\hat{\theta})}\,.$$

where $Z_{a/2}$ is the upper percentile of standard normal variate

Method of Least-Square Estimation (LSE)

The LSE of the unknown parameters α, λ and θ of $EGEG(\alpha,\beta,\lambda,\theta)$ distribution can be obtained by using the principle of optimization. Here, we have estimated parameters by minimizing

$$A(x;\alpha,\beta,\lambda,\theta)=\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^2 \quad (18)$$

with respect to unknown parameters α, β, λ and θ.

Suppose $F(X_i)$ denotes the CDF of the ordered random variables $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ where $\{X_1, X_2, \ldots, X_n\}$ is a random sample of size n taken from a distribution function F(.). The least-square estimators of α, λ and θ say $\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \ and \ \hat{\theta}$ respectively, can be obtained by minimizing

$$A(x;\alpha,\beta,\lambda,\theta)=\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{i}{n+1}\right]^2 ;\ x>0,(\alpha,\lambda,\theta)>0 \quad (19)$$

with respect to α, β, λ and θ.

Differentiating (19) with respect to α, λ and θ we get,

$$\frac{\partial A}{\partial\alpha}=2\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{i}{n+1}\right]\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}$$

$$\ln\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]$$

$$\frac{\partial A}{\partial\beta}=2\alpha\theta\lambda\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{i}{n+1}\right]\frac{x e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}^2}$$

$$\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha-1}\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda-1}$$

$$\frac{\partial A}{\partial \lambda}=-2\alpha\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{i}{n+1}\right]\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha-1}$$

$$\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\ln\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}$$

$$\frac{\partial A}{\partial \theta}=-2\alpha\lambda\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{i}{n+1}\right]\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha-1}$$

$$\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda-1}\frac{e^{-\beta x_{(i)}}(1-e^{-\beta x_{(i)}})}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}^{2}}$$

In similar manner, we can estimate the weighted least square estimators by minimizing

$$D(X;\alpha,\beta,\lambda,\theta)=\sum_{i=1}^{n}w_i\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{i}{n+1}\right]^{2}$$

with respect to α, β, λ and θ. The weights w_i are computed as $w_i=\frac{1}{Var(X_{(i)})}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$

Hence, the weighted least square estimators of α, β, λ and θ respectively can be obtained by minimizing,

$$D\left(X;\alpha,\beta,\lambda,\theta\right)=\sum_{i=1}^{n}\frac{\left(n+1\right)^{2}\left(n+2\right)}{i\left(n-i+1\right)}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-\left(1-\theta\right)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{i}{n+1}\right]^{2} \tag{20}$$

Method of Cramer – Von-Mises estimation (CVME)

The Cramer-Von-Mises estimators of α, β, λ and θ are obtained by minimizing the function

$$Z\left(X;\alpha,\beta,\lambda,\theta\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\mid\alpha,\beta,\lambda,\theta\right)-\frac{2i-1}{2n}\right]^{2}$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-\left(1-\theta\right)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{2i-1}{2n}\right]^{2} \tag{21}$$

Differentiating (21) with respect to α, λ and θ we get,

$$\frac{\partial Z}{\partial\alpha}=2\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-\left(1-\theta\right)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{2i-1}{2n}\right]\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-\left(1-\theta\right)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}$$

$$\ln\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-\left(1-\theta\right)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]$$

$$\frac{\partial Z}{\partial\beta}=2\alpha\theta\lambda\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-\left(1-\theta\right)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{2i-1}{2n}\right]\frac{xe^{-\beta x_{(i)}}}{\left\{1-\left(1-\theta\right)e^{-\beta x_{(i)}}\right\}^{2}}$$

$$\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha-1}\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda-1}$$

$$\frac{\partial Z}{\partial \lambda}=-2\alpha\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{2i-1}{2n}\right]\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha-1}$$

$$\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\ln\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}$$

$$\frac{\partial A}{\partial \theta}=-2\alpha\lambda\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha}-\frac{2i-1}{2n}\right]\left[1-\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda}\right]^{\alpha-1}$$

$$\left\{\frac{\theta e^{-\beta x_{(i)}}}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}}\right\}^{\lambda-1}\frac{e^{-\beta x_{(i)}}(1-e^{-\beta x_{(i)}})}{\left\{1-(1-\theta)e^{-\beta x_{(i)}}\right\}^{2}}$$

By solving $\frac{\partial Z}{\partial \alpha}=0, \frac{\partial Z}{\partial \beta}=0, \frac{\partial Z}{\partial \lambda}=0 \ and \ \frac{\partial Z}{\partial \theta}=0$ simultaneously we obtain the CVM estimators.

Data analysis and application

This section represents the analysis of real data set to verify the proposed model. The data set is data purposed by Hinkley (1997) given as

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47,1.43, 3.37, 2.20, 3, 3.09, 1.51, 2.10, 0.52,1.62,1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05

Exploratory data Analysis

The objective of data analysis is to obtain requiredinformation from the data set used. The exploratory data analysis has been included in the modern statistical data analysis tools. Exploratory data analysis is a group of methods to display and to summarize the data:

i. Displaying the data in graph which shows overall patterns and unusual observations (Boxplot, Histogram, Density curve e.t.c.)

ii. Computing descriptive statistics which summarize specific aspects of the data (Center, Spread, Skewnessand Kurtosis e.t.c.)

We have applied basic EDA techniques for above dataset and the results are presented in table 2.

Table 2

Summary statistics

Min.	1st Qu.	Median	Mean	3rd Qu.	SD	Skewness	Kurtosis	Max.
0.320	0.915	1.470	1.675	2.087	1.000616	1.086682	4.206884	4.750

The Boxplot and Total Time Test (TTT) plot of above data are displayed in figure 3. To check whether or not our particular model can be applied to adata set, Total Time Test (TTT) plot has been used.Since, the TTT plot of the data is concave which indicates that increasing the hazard rate shape of the proposed distribution.

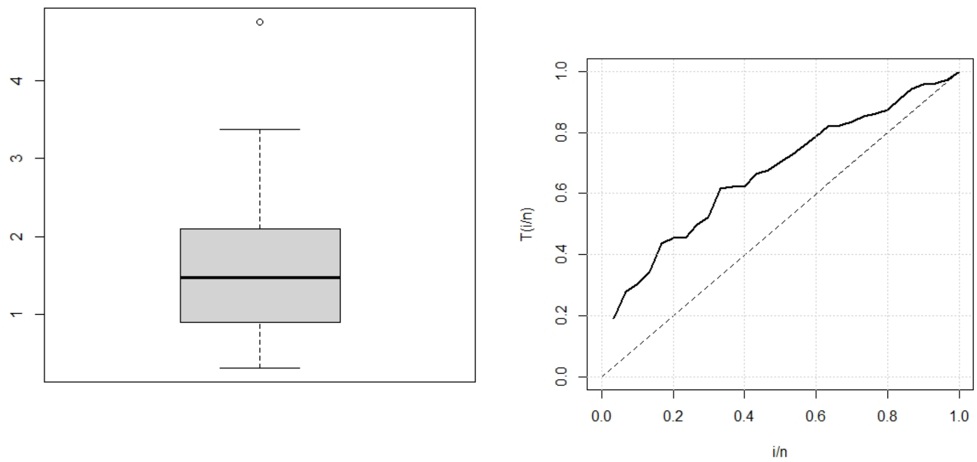

Figure 3: Boxplot (left panel) and TTT plot (right panel) for EGEG

In Figure 4, we have plotted the Q-Q plot and P-P plot and it is observed that the EGEG distribution fits the data very well.

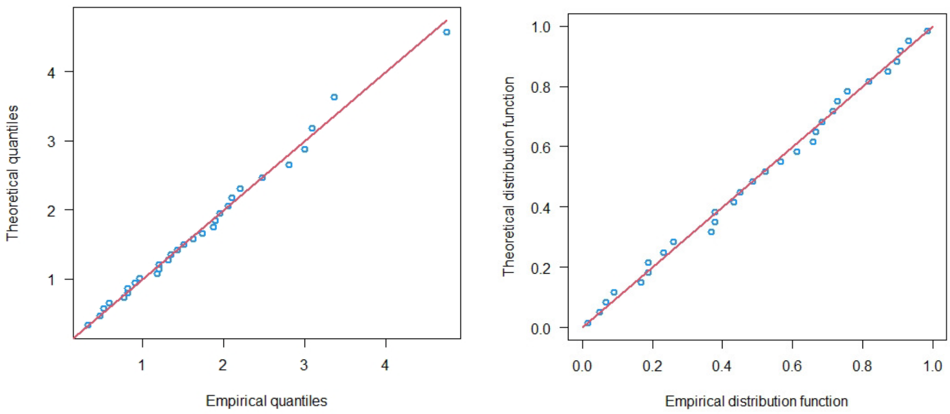

Figure 4: Q-Q plot (left panel) and P-P plot (right panel) of EGEG

Parameter estimation

By employing the optima () function in R software (R Core Team, 2020) and (Ming Hui, 2019), we have calculated the MLEs of EGEG distribution by maximizing the likelihood function (16). We have obtained the value of Log-Likelihood is l = -85.205. In Table 3 we have demonstrated the MLE's with their standard errors (SE) for α, β, λ and θ.

Table 3:

Maximum likelihood estimator and Standard Error

Parameters	MLE	Standard Error
α	15.3245	61.495855
β	2.04140	1.9875511
λ	0.57719	0.4707032
θ	0.08461	0.5245281

In Table 4 we have presented the estimated value of the parameters of EGEG distribution using MLE, LSE and CVE method and their corresponding negative log-likelihood, AIC,BIC and KS statistics with p-value.

Table 4

Estimated parameters, log-likelihood, AIC, BIC and KS statistic

Methods	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	LL	AIC	BIC	*KS(p-value)*
MLE	15.3245	2.0414	0.5772	0.0846	-37.973	83.946	89.551	0.9987 (0.069585)
LSE	12.9210	3.31511	0.3600	0.0165	-39.836	87.672	93.276	0.3257 (0.173720)
CVE	15.9035	3.3301	0.3644	0.0244	-38.162	84.322	89.927	0.9999 (0.060332)

Figure 5 is the histogram and density plot curve for different method of the estimation. It also compares the empirical cumulative distribution curve and the fitted distribution function curves using different methods of estimation is Density curves fit well for proposed model EGEG.

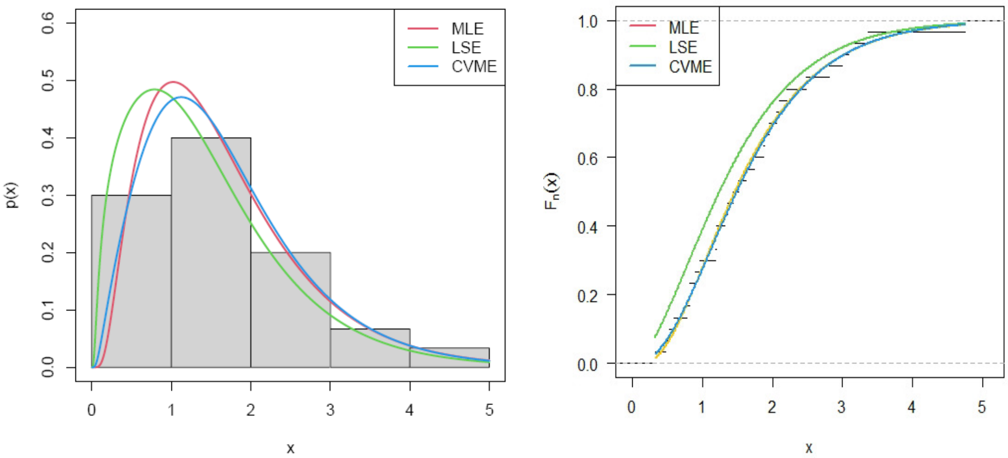

Figure 5: Histogram vs density plot (left panel) and ecdf vs fitted cdf (right panel) of EGEG

Parameters of the proposed model EGEG along with the different alternative models named Half Logistic Nadarajah Haghighi (HLNHE) Distribution (Joshi & Kumar, 2020), Generalized Inverted Generalized Exponential (GIGE) (Oguntunde et al.,2014), A weighted Inverted Exponential Distribution (WIED) (Hussian, 2013), Exponentiated Inverted Weibull Distribution (EIWD) (Flaih et al., 2012) and Logistic Inverse exponential (LIE) (Chaudhary and Kumar,2020) are tabulate in table 5.

(i). Half Logistic Nadarajah Haghighi (HLNHE) Distribution

$$f_{HLNHE}(x)=\frac{2\alpha\beta\lambda(1+\alpha x)^{(\beta-1)}\,exp\left(\lambda\left(1-(1+\alpha x)^{\beta}\right)\right)}{\left[1+exp\left(\lambda\left(1-(1+\alpha x)^{\beta}\right)\right)\right]^{2}}\quad;\alpha,\beta,\lambda>0,x>0$$

(ii). A Weighted Inverted Exponential Distribution (WIED)

$$f(t;\lambda,\alpha)=(1+\alpha)\frac{\lambda}{t^{2}}e^{\frac{-\lambda}{t}}(1-e^{\frac{-\lambda}{\alpha t}}),t>0,\alpha>0,\lambda>0$$

(iii). Exponentiated Inverted Weibull Distribution (EIWD)

$$f(x;\theta,\beta)=\theta\beta x^{-(\beta+1)}(e^{-x-\beta})^{\theta},x>0,\theta>0,\beta>0$$

(iv). Generalized Inverted Generalized Exponential (GIGE)

$$f_{GIGE}(x)=\alpha\lambda\gamma x^{-2}e^{-\gamma(\lambda/x)};\left(1-e^{-\gamma(\lambda/x)}\right)^{\alpha-1}\;;\alpha,\lambda,\gamma>0\,,x>0,$$

(v). Logistic inverse Exponential (LIE) distribution

$$f_{LIE}(x)=\frac{\alpha\lambda}{x^{2}}\frac{\exp\{\lambda/x\}\left[\exp\{\lambda/x\}-1\right]^{\alpha-1}}{\left[1+\{\exp(\lambda/x)-1\}^{\alpha}\right]^{2}};(\alpha,\lambda)>0,x>0$$

Table 5

Estimated parameter values of different models

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$
EGEG	15.3245 (61.4959)	2.0414 (1.9876)	0.5772(0.4707)	0.0846(0.5245)	
WIED	2.6782 (17.6573)		2.2221(0.4330)		
EIWD		1.5496 (0.2026)		1.02520(0.1978)	
HLNHE	26.818 (18.2724)	1.5258 (0.2273)	0.003611(0.0013)		
GIGE	3.3196 (1.0657)		9.8260 (96.5594)		0.22614(2.2220)
LIE	1.8792 (0.2906)		0.94534(0.1102)		

Model Comparison

Here, Akaike's information criteria(AIC),Bayesian information criteria(BIC),Corrected Akaike's information criteria, Hannan-Quinn Information Criteria(HQIC),log-likelihood values, KS values along with p-values are calculated and tabulated in table 6. Model having lowest value of AIC, BIC, CIAC and HQIC is considered as the best model among the models taken in consideration. Here proposed model EGEG has least values so our model is better as compared to the above models.

Table 6

AIC, BIC, CIAC, HQIC, Ks values (p-values) of different models

Models	AIC	BIC	CAIC	HQIC	-LL
EGEG	83.94566	89.55045	85.54566	80.84217	37.9728
WIED	85.66180	88.46419	86.10624	84.11006	40.8309
EIWD	87.83400	90.63639	88.27844	86.28226	41.9170
HLNHE	84.65766	88.86125	85.58074	82.33004	39.12883
GIGE	85.31920	89.52279	86.24228	82.99158	39.65960
LIE	86.11962	90.32321	87.04270	83.79200	40.05981

The Histogram versus the density function of fitted distributions and Empirical distribution function versus estimated distribution function of EGEG distribution and some selected distributions are presented in Figure 5. From the graph, it is clear that the proposed model IEOLE fits better than the other considered models.

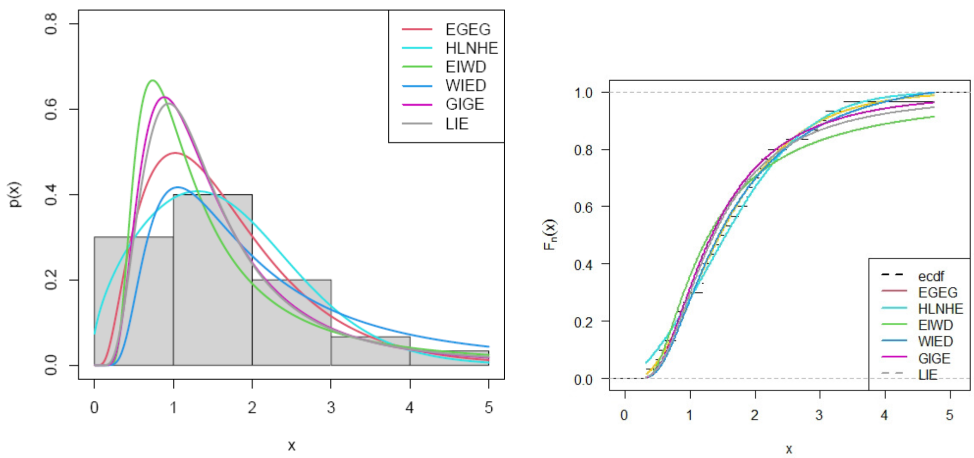

Figure 6: Histogram vs fitted pdf (left panel) and ecdf vs cdf of different models.

It is found that the proposed model EGEG has a good fit of theoretical distribution versus empirical distribution in both plots.

One sample Kolmogorov - Smirnov test is applied for the validation. The K-S test (D = 0.06958, p-value = 0.9987) is not statistically significant which suggest that it fits well with the theoretical distribution versus empirical distribution (p-value = 0.9987). Empirical distribution function and fitted distribution functions are shown in figure 6. It is clear that the proposed model provides the satisfactory fit to the given data.

We also compared the empirical distribution CDF with the estimated cumulative distribution function CDF of proposed model along with other models. Here we compared the estimated pdf of the proposed model EGEG with estimated pdf of other models on same set of the data. The proposed model gives the better fit in both the density fit and empirical cumulative distribution function than other models.

Conclusion

This paper named as Exponentiated Generalized Exponential Geometric Distribution (EGEG) is based on the Exponentiated Generalized (EG) class of distribution and Exponential Geometric distribution. This distribution contains four parameters and is uni-model, positively skewed and leptokurtic. Hazard rate function is inverted bathtub with decreasing order. Here we have included explicit expression for hazard rate function, likelihood function, quantile function and survival function. Parameters of new distribution EGEG are estimated by using maximum likelihood estimate and then confidence intervals for all parameters are determined. Here we have compared the PDF and CDF of proposed model with other five models. Proposed model (EGEG) is also compared with other models through different criteria like K-S, Anderson Darling and Cramer's von Mises test, P-P plot and Q-Q plot showing that proposed model has great flexibility than models taken in comparison. We found that the proposed model is more suitable for real lifetime data analysis and modeling.

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